SLOW MOTIONS OF A LIQUID DROP IN A VISCOUS LIQUID

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The problem of the fully established slow translational motion of a round drop (bubble) in a viscous liquid was solved by Adamar and Rybchinskii [1, 2]. The results of experimental measurements are rarely in agreement with the Adamar–Rybchinskii formula. This is connected with braking of the flow due to surface-active impurities, which are usually rather numerous in liquids. Nevertheless, we shall consider the problem of the not fully established motion of a drop in the simplest case, assuming that there are no surface-active substances. The article discusses problems of the vibrations and motions of a spherical drop in a viscous liquid, with arbitrary accelerations. An analysis is made of a formula for the force of resistance of a drop of liquid with a high viscosity, an elastoviscous drop, and a particle with "slipping-through."

1. If the motions of a drop are so slow that they can be limited by a linear approximation, and the drop is so small that a change in its form can be neglected [1, 2], then the flows inside and outside the drop are described by the following equations (we assume the liquids to be incompressible):

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla p + \eta \Delta \mathbf{u}, \quad \nabla \mathbf{u} = 0; \tag{1.1}$$

here the liquid outside the drop is characterized by the density ρ and the viscosity η, and the liquid inside the drop by ρ' and η' (the characteristics of the inner region are denoted by primes).

Assuming that the law of change in the velocity of the drop* $\mathbf{U}(t)$ is known, we shall seek the law of change in the force of the resistance $\mathbf{F}(t)$ acting on the drop. Here the liquid far from the drop is assumed to be motionless. Using the linearity of the problem, and a Fourier transform with respect to the time $f(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} f(\mathbf{r}, t)\, e^{i\omega t} dt$, we first go over to the simpler problem of the resistance encountered by a drop whose velocity varies harmonically, and then return to the general problem, using an inverse Fourier transform,

$$f(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\mathbf{r}, \omega)\, e^{-i\omega t} d\omega. \tag{1.2}$$

For the complex amplitudes of the pressures $p(\mathbf{r}, \omega)$ and $p'(\mathbf{r}, \omega)$ and velocities $\mathbf{u}(\mathbf{r}, \omega)$ and $\mathbf{u}'(\mathbf{r}, \omega)$ arising outside and inside a drop whose velocity varies according to the law $\mathbf{U}(\omega)e^{-i\omega t}$, from Eqs. (1.1) it follows that

$$-i\omega\rho \mathbf{u}(\mathbf{r}, \omega) = -\nabla p(\mathbf{r}, \omega) + \eta \Delta \mathbf{u}(\mathbf{r}, \omega), \quad \nabla \mathbf{u}(\mathbf{r}, \omega) = 0. \tag{1.3}$$

Analogous equations are obtained for the characteristics of the inner region (with a complete analogy, the relationships for them are not written separately below).

Since the solutions for $\mathbf{u}$ and p can depend only on the vectors $\mathbf{r}$, $\mathbf{U}(\omega)$, linearly on $\mathbf{U}$, their general form will then be

*Thanks to the linearity of the problem and the spherical nature of the drop, the translational and rotational motions of the drop can always be considered separately. In what follows, only the translational motion is analyzed.

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$$p(\mathbf{r},\ \omega)=p_0+\mathbf{nU}\varphi_0(r),\ r\equiv|\mathbf{r}|;$$

$$\mathbf{u}(\mathbf{r},\ \omega)=\mathbf{n}(\mathbf{nU})\varphi_1(r)+\mathbf{m}\varphi_2(r),\ \mathbf{m}\equiv\mathbf{U}-\mathbf{n}(\mathbf{nU}). \tag{1.4}$$

The convenience of an expansion in terms of the above form is connected with the property of orthogonality $\mathbf{nm}=0$. Substituting these expressions into Eqs. (1.3), we find

$$\begin{gathered}\frac{\partial\varphi_1}{\partial r}+\frac{2}{r}(\varphi_1-\varphi_2)=0;\\ \frac{\partial}{\partial r}(\eta\Delta+i\omega\rho)\left(r\frac{\partial\varphi_1}{\partial r}+3\varphi_1\right)=0;\\ \varphi_0/r=i\omega\rho\varphi_2+\eta\left[\frac{\partial^2\varphi_2}{\partial r^2}+\frac{2}{r}\frac{\partial\varphi_2}{\partial r}+\frac{2}{r^2}(\varphi_1-\varphi_2)\right].\end{gathered} \tag{1.5}$$

We obtain the general solution of the second equation of system (1.5):

$$\varphi_1=c_0+c_1\frac{1}{r^3}+c_2\frac{1+\lambda r}{\lambda^2r^3}e^{-\lambda r}+c_3\frac{1-\lambda r}{\lambda^2r^3}e^{\lambda r},\ \lambda=(1-i)\sqrt{\frac{\rho\omega}{2\eta}},\quad c_k=\text{const}.$$

From the condition of the boundedness of the solution with $r=0$ it follows that $c_1=0$, $c_2=c_3$; therefore, the flow of liquid inside the drop is represented by the equations

$$\varphi_1'(r)=a_1+a_2\frac{\gamma(\lambda'r)}{\lambda'^2r^3};\quad\gamma(x)\equiv x\,\mathrm{ch}\,x-\mathrm{sh}\,x. \tag{1.6}$$

The condition of immobility of the liquid far from the drop leads to $c_0=c_3=0$, so that outside the drop

$$\varphi_1(r)=\frac{b_1}{r^3}+b_2\frac{1+\lambda r}{\lambda^2r^3}e^{-\lambda r}. \tag{1.7}$$

From the first equation of system (1.5) and relationships (1.6), (1.7) it follows that

$$2\varphi_2'(r)+\varphi_1'(r)=3a_1+a_2\frac{\mathrm{sh}\,\lambda'r}{r};\quad 2\varphi_2(r)+\varphi_1(r)=-b_2\frac{e^{-\lambda r}}{r}. \tag{1.8}$$

In turn, these formulas permit finding the values from the third equation of system (1.5),

$$\varphi_0'(r)=i\omega\rho r a_1;\quad\varphi_0(r)=-\frac{i\omega\rho}{2r^2}b_1. \tag{1.9}$$

Thus, solutions are found with an accuracy up to the four undetermined coefficients a_1, a_2, b_1, b_2, which are found from the boundary conditions at the spherical surface of the drop (we neglect its deformation).

The conditions of the equality of the normal and tangential components of the velocity of the surface of the drop give (a is the radius of the drop)

$$\varphi_1'(a)=\varphi_1(a)=1;\quad\varphi_2'(a)=\varphi_2(a). \tag{1.10}$$

As a fourth condition we use the equality of the tangential forces (under these circumstances the normal forces are not equal and the deformation of the drop is not taken into consideration here). Solutions of the form (1.4) correspond to the general formula for the stresses

$$\sigma_{ij}=\delta_{ij}\left[-p_0+\left(2\eta\frac{\varphi_1-\varphi_2}{r}-\varphi_0\right)\mathbf{nU}\right]+n_in_j2\eta\left(\frac{\partial\varphi_1}{\partial r}-\frac{\varphi_1-\varphi_2}{r}\right)\mathbf{nU}-(n_im_j+n_jm_i)\eta\left(\frac{\partial\varphi_2}{\partial r}+\frac{\varphi_1-\varphi_2}{r}\right), \tag{1.11}$$

which, using the first equation of system (1.5), permits writing the condition for the equality of the shear stresses on both sides of the surface in the form

$$\eta\frac{\partial}{\partial r}(2\varphi_2-\varphi_1)|_{r=a}=\eta'\frac{\partial}{\partial r}(2\varphi_2'-\varphi_1')|_{r=a}. \tag{1.12}$$

Substituting the solutions into conditions (1.10), (1.12), we obtain

$$\frac{a_2}{a}=\frac{-3\eta(1+\lambda a)}{\eta(3+\lambda a)\delta(\lambda'a)+\eta'[\gamma(\lambda'a)-2\delta(\lambda'a)]}; \tag{1.13}$$

$$a_1=1-\frac{a_2}{a}\frac{\gamma(\lambda'a)}{(\lambda'a)^2},\quad\delta(x)\equiv\mathrm{sh}\,x-3\gamma(x)/x^2; \tag{1.14}$$

$$-\frac{b_2}{a}e^{-\lambda a}=3+\frac{a_2}{a}\delta(\lambda'a);\quad\frac{b_1}{a^3}=\frac{\lambda^2a^2+3\lambda a+3}{\lambda^2a^2}+\frac{1+\lambda a}{\lambda^2a^2}\frac{a_2}{a}\delta(\lambda'a). \tag{1.15}$$

To find the complex amplitude of the force of the resistance acting on a vibrating drop, we must integrate the stress from (1.11) over its surface. For a spherical drop, this operation reduces to an easily performed averaging over the directions

$$F_i(\omega) = \int\limits_{r=a} \sigma_{i\alpha}(\mathbf{r}, \omega) n_\alpha dS = 4\pi a^2 \langle \sigma_{i\alpha}(\mathbf{r}, \omega) n_\alpha \rangle|_{r=a} = \frac{4\pi}{3} a^2 \left[-\varphi_0(a) + 2\eta \left(\frac{\partial \varphi_2}{\partial r} + \frac{\partial \varphi_1}{\partial r} + \frac{\varphi_1 - \varphi_2}{r} \right) \Big|_{r=a} \right] U_i(\omega).$$

The latter expression is simplified using (1.7)-(1.9),

$$F_i(\omega) = -2\pi a\eta \left[\frac{\lambda^2 a^2}{3} - (1 + \lambda a) \frac{b_2}{a} e^{-\lambda a} \right] U_i(\omega). \tag{1.16}$$

In the partial case of a rigid sphere ($\eta' \gg 1$), in accordance with (1.13), (1.15), $b_2 e^{-\lambda a} = -3a$ and the formula for the force assumes a well-known form [2]. In the case of steady-state motion, $(\omega, \lambda, \lambda' \to 0)$ — $b_2 = a[(2\eta + 3\eta')/(\eta + \eta')]$ and formula (1.16) goes over to the Adamar–Rybchinskii formula.

Let us consider the simplifications arising with vibrations of low frequency $\omega \ll \eta'/(\rho' a^2)$. With such frequencies, it is sufficient to leave only a few terms in the expansions:

$$\gamma(x) = \sum_{k=0}^{\infty} \frac{1}{2k+3} \frac{x^{2k+3}}{(2k+1)!}; \quad \delta(x) = \sum_{k=0}^{\infty} \frac{1}{(2k+5)(2k+3)} \frac{x^{2k+3}}{(2k+1)!}.$$

Then from relationships (1.13)-(1.15) we obtain

$$-\frac{b^2}{a} e^{-\lambda a} \approx \frac{2\eta + 3\eta'}{\eta + \eta' + \lambda a\eta/3} + \frac{\eta\eta'(1 + \lambda a)}{(\eta + \eta' + \lambda a\eta/3)^2} \frac{(\lambda' a)^2}{21} + \ldots$$

We note that the smallness of $|\lambda' a|$ still does not mean the smallness of $|\lambda a|$. With $\eta'\rho' \gg \eta/\rho$, i.e., for a liquid which is very viscous in comparison with the surrounding liquid, within the framework of the approximation under consideration, $\omega \gg \eta/(\rho a^2)$. Discarding all terms with λ', for the amplitude of the resistance force we obtain the simple formula

$$F_i(\omega) = -2\pi a \left[-\frac{i\omega\rho a^2}{3} + \eta \frac{(2\eta + 3\eta')(1 + \lambda a)}{\eta + \eta' + \lambda a\eta/3} \right] U_i(\omega). \tag{1.17}$$

2. To determine the law of change of the resistance force $\mathbf{F}(t)$ with an arbitrary change in the velocity of the drop $\mathbf{U}(t)$, a summation of the type (1.2) must be carried out in (1.16) over all frequencies. After this, the general formula assumes the form

$$\mathbf{F}(t) = -\frac{2\pi}{3} \rho a^3 \frac{d\mathbf{U}}{dt} - 2\pi a\eta \frac{2\eta + 3\eta'}{\eta + \eta'} \mathbf{U}(t) - 2\pi a\eta \int\limits_{-\infty}^{t} K(t - t') \frac{d\mathbf{U}}{dt'} dt'$$

with the memory function

$$K(t) = -\frac{\rho}{2\pi\eta} \int\limits_{-\infty}^{\infty} \left(\frac{1 + \lambda a}{\lambda^2 a} b_2 e^{-\lambda a} + \frac{2\eta + 3\eta'}{\eta + \eta'} \frac{1}{\lambda^2} \right) e^{-i\omega t} d\omega. \tag{2.1}$$

Let us find the explicit form of $K(t)$ with large times. Since, in the frequency expansion (2.1) with large values of t the principal contribution is made by small frequencies, with $t \gg a^2\rho'/\eta'$, terms with $\lambda' a$ can generally be omitted from the expression for b_2. By the same token, the problem reduces to the application of an inverse Fourier transform in (1.17),

$$K(t) \approx \frac{a\rho\alpha}{6\pi} \int\limits_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\lambda(\eta + \eta' + \lambda a\eta/3)} d\omega = \alpha e^T \operatorname{Erf} \sqrt{T},$$
$$\alpha \equiv \frac{(2\eta + 3\eta')^2}{\eta(\eta + \eta')}, \quad T \equiv \frac{9(\eta + \eta')^2}{\rho\eta a^2} t. \tag{2.2}$$

In the final stage of the acceleration of a drop (with $T \gg 1$), an asymptotic expansion of the probability integral of the errors permits writing

$$K(t) \approx \frac{\alpha}{\sqrt{\pi T}} + \ldots, \quad t \gg a^2\rho/\eta,\ a^2\rho'/\eta',$$

and the formula for the resistance force of a drop assumes a form analogous to the formula for the resistance of a solid sphere [2].

In a description of the Brownian movement of particles, together with the conditions for the adhesion of the liquid on the surface of the particles, conditions for slipping-through are also taken into account. The latter corresponds to a degenerate case of the problem under consideration with $\eta' = 0$. The boundary conditions of the absence of tangential forces at the surface of a spherical particle, and of penetration of liquid through its surface

$$\frac{\partial}{\partial r}(2\varphi_2 - \varphi_1)|_{r=a} = 0, \quad \varphi_1(a) = 1$$

lead to a simple expression for b_2:

$$\frac{b_2}{a} e^{-\lambda a} = -\frac{6}{3+\lambda a}.$$

Under these circumstances, for the amplitude of the force and the memory function we find the expressions

$$\mathbf{F}(\omega) = -2\pi a\eta\left(\frac{\lambda^2 a^2}{3} + 6\frac{1+\lambda a}{3+\lambda a}\right)\mathbf{U}(\omega); \quad K(t) = 4e^{\tau}\,\mathrm{Erf}\sqrt{\tau}, \quad \tau \equiv \frac{9\eta}{\rho a^2}t.$$

These formulas follow also from (1.17), (2.2), with $\eta' = 0$ but, in distinction from them, are formally suitable with any arbitrary frequencies and times.

Let us now analyze the behavior of a drop of an elastoviscous liquid in the final stage of its acceleration in a viscous liquid ($t \gg a^2\rho/\eta, a^2\rho'/|\eta'|$). The transition from the vibrations of a drop of a viscous liquid with the viscosity coefficient η' to the vibrations of a drop of an elastoviscous liquid reduces, in a linear approximation, to the replacement of the coefficient η' by the frequency function $\eta'_+(\omega)$. In a liquid with one relaxation time $\eta'_+(\omega) = \eta'/(1 - i\theta\omega)$.

Neglecting for the sake of simplicity all terms with $\lambda' a$ and λa, we obtain

$$\mathbf{F}(\omega) \approx -2\pi a\eta\frac{2\eta + 3\eta'_+(\omega)}{\eta + \eta'_+(\omega)}\mathbf{U}(\omega).$$

In the case of a liquid with one relaxation time, an inverse Fourier transform gives

$$\mathbf{F}(t) \approx -2\pi a\eta\frac{2\eta + 3\eta'_0}{\eta + \eta'_0}\mathbf{U}(t) - 2\pi a\eta\int_{-\infty}^{t} K(t-t')\frac{d\mathbf{U}}{dt'}dt';$$

$$K(t) = -\frac{\eta'_0}{\eta + \eta'_0}e^{-\left(1+\frac{\eta'_0}{\eta}\right)\frac{t}{\theta}}, \quad \eta'_0 \equiv \eta'_+(0).$$

Thus, the characteristic memory time of a drop is found to be $1 + \eta'_0/\eta$ less than the relaxation time of the liquid of which it is made up.

LITERATURE CITED

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